LETTER TO THE EDITOR

Spin block persistence at finite temperature

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Abstract. We explore a new definition of the persistence exponent, measuring the probability that a spin never flips after a quench of an Ising-like model at a temperature $0 < T < T_c$, while the usual definition only makes sense at $T = 0$. This probability is now defined for spin blocks, and a general scaling for it, involving time and block linear size is introduced and illustrated by extensive simulations.

Recently there has been significant progress in the study of coarsening systems [1]. Our understanding of phase-ordering phenomena is now structured by simple ideas such as dynamic scaling and universality. For instance, it is well established that nonconserved order-parameter dynamics are characterized by a single length scale $L(t) \sim t^{1/z}$, and that a nontrivial exponent λ appears in the scaling of the order-parameter correlation function, $\langle \varphi(\boldsymbol{x}, t')\varphi(\boldsymbol{x}, t) \rangle \sim [L(t)/L(t')]^\lambda$, for $t' \gg t$. However, advances in this field have been to a large extent boosted by progress in numerical simulations of lattice systems such as the Ising model, while few analytical results are known except in one dimension. Therefore surprises are still to be expected as one probes more and more complicated correlations.

Such a surprise came out recently as much interest was devoted to the study of the so-called *persistence* probability [2–4]. Consider the following simple question: In a simulation of the Glauber dynamics of the Ising model at zero temperature, what is the fraction of spins $p(t)$ which have never flipped since the initial time? It turns out that p exhibits a nontrivial algebraic decay $p(t) \sim t^{-\theta}$. A quantity such as $p(t)$ involves the whole history of the system and is not easy to study analytically. Derrida *et al* [5] showed analytically that $\theta = \frac{3}{8}$ in one dimension, but in higher dimensions, θ could only be determined by numerical simulations [2, 3] or approximate methods [6–8]. More generally, the probability that a stochastic physical quantity has never changed sign since the origin of its evolution arises naturally in the context of nonequilibrium systems. Even for simple scalar diffusion with zero mean random initial conditions, a nontrivial algebraic decay is found [7, 8].

For ferromagnetic systems, until very recently, persistence had only been defined and studied at zero temperature for a single spin (local order parameter) [2, 3, 5, 6, 8] or at T_c for the total magnetization (global order parameter), where it yields a new independent critical exponent [9]. For $T > 0$, $p(t)$ decays exponentially due to thermal fluctuations, and the $T = 0$ definition does not look very promising. However, there are good reasons to be interested in a definition at a *finite temperature* $0 < T < T_c$. First, numerics, renormalization group arguments or large-N calculations [1] assess that finite (noncritical) temperature correlations have the same scaling as zero-temperature correlations, with the

same λ. Thus, it is natural to expect the same kind of universality to hold for the persistence exponent, and it is worthwhile checking this point. In addition, some discrepancies were found between the value of θ for discrete (Ising) and continuous (φ^4) models [6]. These discrepancies were attributed to anisotropy effects in lattice systems at zero temperature, which should be lowered at finite T. Thirdly, certain important models such as the Ising model with conserved Kawasaki dynamics do not coarsen at zero temperature, due to finite-energy barriers, and must be simulated at $T > 0$. However, the corresponding continuous model (model B) exhibits coarsening at zero temperature, and thus one needs a rule to extrapolate information from finite-temperature simulations using Kawasaki dynamics.

In a very recent paper, Derrida [10] proposed studying persistence at finite temperature for nonconserved Ising and Potts models by comparing two systems, A and B, evolving with *the same thermal noise* from two different initial conditions: completely random for A, and all spins equal to $+1$ for B (fundamental state). The idea is that B experiences flips solely due to thermal fluctuations in an ordered system. Thus a flip is only recorded when a spin at the same site in both samples does not flip simultaneously in A and B. In two dimensions Derrida found that the corresponding persistence probability $p(t)$ decays algebraically with an exponent close to the value of θ at $T = 0$. More extensive simulations performed by Stauffer [11] also suggest a temperature-independent exponent equal to θ in two dimensions, but significantly different in three and four dimensions. The value found in three dimensions for $T > 0$ is in good agreement with an approximate continuous theory at $T = 0$ [6] (see conclusion). Derrida's method is ingenious and straightforwardly implemented, but it cannot be used to study conserved dynamics, as system B would not evolve with Kawasaki dynamics, and it is not easy to generalize to a continuous field.

In this letter, we propose a very natural method to study persistence at finite temperature, namely 'block scaling', which can be directly performed on a single sample. The idea stems from *à la Kadanoff* renormalization group ideas. At finite temperature, we consider the persistence of coarse-grained spin variables obtained by integrating the order parameter (spin) on blocks. When the size l of the blocks is increased, the effective temperature flows to zero, which establishes a connection with the zero-temperature dynamics. It is clear that this definition also applies to continuous models. We shall restrict ourselves to nonconserved ferromagnetic models (model A) with $L(t) \propto \sqrt{t}$, but the same method can be used to study conserved models.

Before considering finite temperature, it is instructive to see how block scaling works at $T = 0$, for which two persistence exponents can be defined: θ for a single spin (local order parameter), and θ_0 corresponding to the probability $p_0(t)$ that the total magnetization (global order parameter) has never changed sign [9, 12]. Majumdar *et al* [9] have shown the exact result $\theta_0 = \frac{1}{4}$ for the Ising model in one dimension. Cornell and Sire [12] performed direct numerical simulations of $p_0(t)$ in two dimensions, by recording the time when the global magnetization first changes sign at $T = 0$. This requires a very large number of runs, which drastically limits the sample size ($L_{\max} \sim 128$). In addition, finite-size scaling is not very conclusive, leading to a large uncertainty on the value of $\theta_0 \approx 0.06 \sim 0.11$.

We now show that block scaling leads to a much easier determination of θ_0 at $T = 0$, before moving to finite T. Let us consider blocks of size l and the probability $p_l(t)$ that the total magnetization of a block has never changed sign since $t = 0$ (we will use blocks with an odd number l^d of spins). For large time, when $L(t) \gg l$, blocks behave as single spins and $p_l(t) \sim c_l t^{-\theta}$, c_l being an increasing function of l, since obviously at large time $p_{l'}(t) > p_l(t)$ if $l' > l$. At early times, when $L(t) \leqslant l$, the system effectively sees infinite blocks, and $p_l(t) \propto t^{-\theta_0}$, where θ_0 is the persistence exponent of the *total magnetization* at $T = 0$. Moreover, in the initial configuration, the larger the blocks the smaller the relative
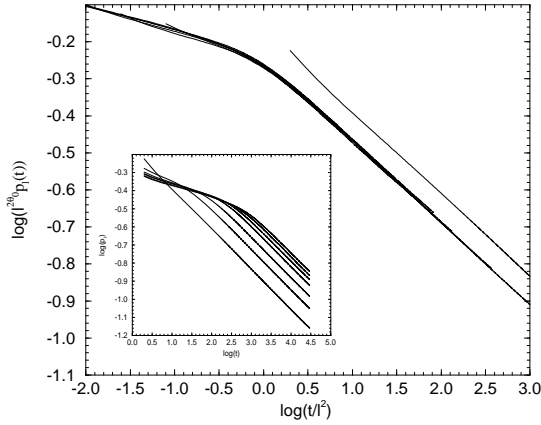
Figure 1. Block persistence at $T = 0$ obtained from simulation of the nonconserved Ising model on a 2000^2 lattice, for $l = 1, 5, 9, 15, 19, 25$, and 31 (from bottom to top in the insert). $p_l(t)$ decays as $t^{-\theta_0}$ at early time and as $t^{-\theta}$ at large time. Excellent scaling is then obtained taking $\theta_0 = 0.09$.
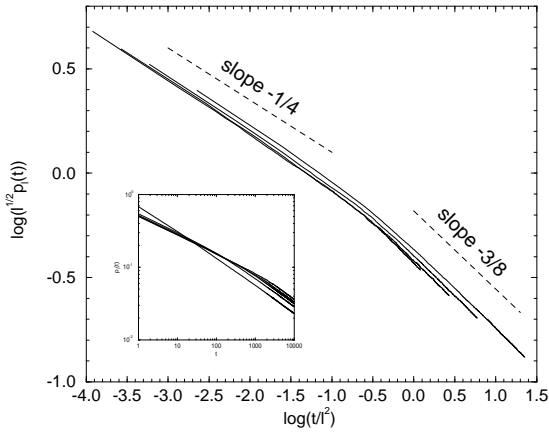

Figure 2. Similar scaling as in figure 1, for a one-dimensional spin chain (200 000 spins, 10 samples), with block size $l = 1, 21, 41, 61, 91$ (from bottom to top in the right part of the insert). $l = 1$ is omitted in the scaling, and the data collapse improves as the block size increases.

fluctuations of the magnetization around zero, therefore $p_{l'}(t) < p_l(t)$, for $l' > l$. The cross over between the two regimes should occur at $t \propto l^2$. These remarks lead us to the following large l scaling,

$$p_l(t) \sim l^{-\alpha} f(t/l^2) \tag{1}$$

where $f(x) \propto x^{-\theta_0}$ when $x \to 0$ and $f(x) \sim x^{-\theta}$ when $x \to \infty$. For finite t, $p_l(t)$ must tend to a finite value for $l \to \infty$, equal to the probability that the global magnetization never changed sign. This requires $\alpha = 2\theta_0$. Hence, computing $p_l(t)$ for several values of l makes it possible to determine θ_0 by adjusting its value to obtain the best data collapse.

To check this scaling, we simulated the $T = 0$ Glauber dynamics for the Ising model in $d = 2$ on a 2000^2 lattice with blocks of linear size $1, 5, 9, 15, 19, 25$, and 31. 20 samples were averaged to obtain the final data presented in figure 1. We find excellent scaling, with $\theta_0 = 0.09$. Similar results were obtained in one dimension, confirming the scaling relation of equation (1) and the theoretical value $\theta_0 = \frac{1}{4}$ (figure 2). Therefore, block scaling is a very convenient and reliable method to determine θ_0.

Now let us move to a finite temperature $0 < T < T_c$ (not too close to T_c, a case studied in a forthcoming paper [13]). The difficulty in defining a persistence exponent comes from the fact that a spin may flip due to thermal fluctuations, leading to an exponential decay $p(t) \sim \exp(-t/\tau)$. Indeed, at $T = 0$, a spin only flips when it is crossed by an interface between a positive and negative domain, whereas at finite temperature, the dominating

process at late time, when the domains are large, is the flip of a spin within a domain due to thermal fluctuations. Therefore, at low temperature, it is natural from classical kinetics intuition to expect an Arrhenius law $\tau \sim \exp(-\Delta\mathcal{E}/T)$, where $\Delta\mathcal{E}$ is the energy barrier to flip a spin (or a block) within an ordered domain. As $T \to 0$, τ diverges and p crosses over to a power law.

It is instructive to justify the Arrhenius law from a random process viewpoint. Let us consider a block of linear size l, and spin block variables φ_l. When $L(t)$ is large enough, the system can be considered locally at equilibrium inside a domain, and, since there are no long-range correlations, the relative fluctuation of φ_l has the scaling $\Delta\varphi_l/\langle\varphi_l\rangle \propto \sqrt{T/l^d}$. Thus $p_l(t)$ is essentially the probability that a stationary random process $X(t)$ with zero mean and mean square fluctuation $\langle X^2 \rangle = T/l^d$ crosses a barrier of amplitude of order 1. If $X(t)$ is Gaussian and Markovian, it is the solution of a simple Langevin equation, with a Gaussian white noise $\eta(t)$ with $\langle\eta(t)\eta(t')\rangle = 2T/l^d\delta(t-t')$. Then it is immediately seen that in exponential time $u = e^t$, $p_l(u)$ is the survival probability of a simple one-dimensional random walker with diffusion coefficient $2T/l^d$, starting from $x = 0$ with a moving absorbing wall at $x(u) \propto \sqrt{u}$. When the amplitude of the fluctuation vanishes, i.e. for small T or large l, this survival probability can be evaluated by using the unperturbed solution of the diffusion equation [14]. At large u, $p_l(u)$ decays with a power law $p_l(u) \propto u^{-\beta}$ and $\beta \propto \sqrt{l^d/T}\exp(-Cl^d/T)$, where C is a constant. Thus, we recover the heuristic Arrhenius law with $\tau \propto 1/\beta$.

The actual stochastic process $\varphi_l(t)$ is certainly non-Markovian. However, for l much bigger than the equilibrium correlation length, it is nearly Gaussian. Moreover, its correlator $C(t) = \langle\varphi_l(t)\varphi_l(0)\rangle$ can be bounded by two Markovian exponential correlators (because there is no long-range correlation in time at equilibrium), and thus the Arrhenius law still holds with proper constants inserted (although the power law in the prefactor may be modified) [15]. The important point is that the effective temperature entering the Arrhenius law of the spin blocks is cut by a factor l^d and that τ diverges very quickly when l is increased, leading to a fast cross over to the $T = 0$ behaviour. For $t \ll \tau$, $p_l(t)$ is expected to behave in the same way as for $T = 0$. Finally, at finite temperature (not too close to T_c, in the vicinity of which a different scaling arises [13]), we expect a scaling of the form $p_l(t) \sim l^{-2\theta_0} f(t/l^2)\exp[-t/\tau(l,T)]$, involving two cross over times clearly visible in figure 3, which shows the result of two-dimensional simulations performed at $T = 2T_c/3$ on a 1000^2 lattice, with $l = 1, 3, 5, 7, 9, 11, 13$. The exponential decay is clearly visible for $l = 1$ and $l = 3$. However, for larger blocks, τ is bigger than the simulation time, and $p_l(t)$ has the $T = 0$ behaviour, with a power law decay with exponent θ fully compatible with the $T = 0$ value ($\theta = 0.22$), for $t \gg l^2$, and a power law decay with exponent θ_0, for $t < l^2$, just as expected. Figure 4 shows the scaling with $\theta_0 = 0.09$ (for $l = 7, 9, 11, 13$, and a slightly smaller $T = T_c/2$ to eliminate the effect of the exponential cut-off).

Thus, block scaling leads to a clear definition of θ at finite temperature as the exponent of the algebraic decay of the scaling function $f(x)$. We find that in two dimensions, the exponents θ and θ_0 do not depend on T and are equal to their $T = 0$ value, in agreement with the results obtained with Derrida's definition. It is also very satisfactory to observe that both scaling functions of figures 1 and 4 are identical up to a multiplying factor, in a very similar way as what is known for the equal-time two-point spin correlation function [1]. It would be worthwhile to use the present method to study the q-state Potts model, as Derrida's data for $q = 7$ suggested a temperature dependence of θ [10]. On the basis of the present work, we would expect θ to be independent of T, but this point has to be confirmed.

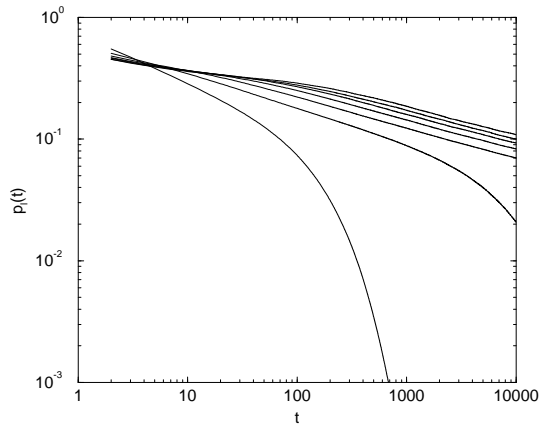We conclude with a look at the puzzling three-dimensional case. Using the present

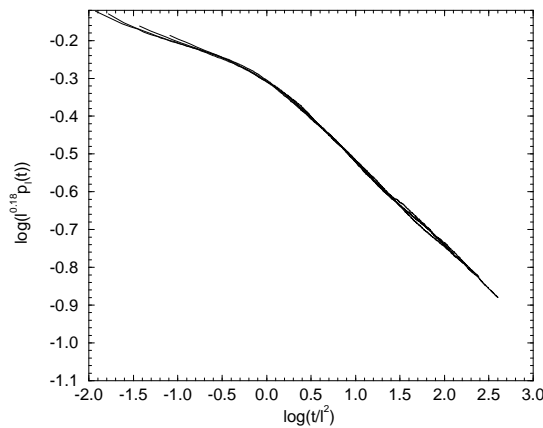**Figure 3.** $p_l(t)$ for $T = 2T_c/3$, and block sizes $l = 1, 3, 5, 7, 9, 11, 13$.

Figure 4. $p_l(t)$ expressed in scaling form for $T = T_c/2$, and block sizes $l = 7, 9, 11, 13$, using the same value for $2\theta_0 = 0.18$ as in the $T = 0$ case. Note the similarity with the $T = 0$ scaling function of figure 1.

block method, we find an exponent $\theta_{T>0}$ consistent with the value $\theta_{T>0} = 0.26$ obtained by Stauffer [11] using Derrida's definition, but different from the $T = 0$ value $\theta_{T=0} = 0.17$ [3, 6]. In fact, it is well known (although a precise explanation is still lacking) [16, 6], that the domain length scale $L(t)$ does not grow as $t^{1/2}$ in three dimensions, but as $t^{0.33}$, presumably due to lattice effects. If we now express our general scaling as a function of the more intrinsic $L(t)$ instead of time itself, we find that for $T = 0$ and $T > 0$ (in the latter case for $l^2 \ll t \ll \tau(l, T)$), both persistence probabilities decay as $p(t) \sim L(t)^{-\theta}$, with the same $\theta \approx \frac{0.17}{0.33} \approx \frac{0.26}{0.5} \approx 0.52$, in good agreement with the theoretical prediction of [6].

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